

# A PROBLEM IN NON-LINEAR DIOPHANTINE APPROXIMATION

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**ABSTRACT.** In this paper we obtain the Lebesgue and Hausdorff measure results for the set of vectors satisfying infinitely many fully non-linear Diophantine inequalities. The set is also associated with a class of linear inhomogeneous partial differential equations whose solubility is related to a certain Diophantine condition. The failure of the Diophantine condition guarantees the existence of a smooth solution.

## 1. INTRODUCTION AND STATEMENTS OF RESULTS

Metric Diophantine approximation of a single linear form is in a first instance concerned with the (Lebesgue and Hausdorff) measure of the set of vectors  $(x_1, \dots, x_n) \in \mathbb{R}^n$  for which there are infinitely many vectors  $(q_1, \dots, q_n, p) \in \mathbb{Z}^{n+1}$  for which the inequality

$$|q_1x_1 + \dots + q_nx_n - p| < \psi(H(\mathbf{q})) \quad (1)$$

is satisfied. Here,  $\psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  denotes a monotonic arithmetic function decreasing to zero and  $H(\mathbf{q})$  denotes the naive height of the vector  $\mathbf{q}$ , i.e.  $H(\mathbf{q}) = \max\{|q_1|, \dots, |q_n|\}$ . The set is well studied. Its Lebesgue measure is given by the famous zero-one law of Groshev [13] and its Hausdorff measure is calculated by Dickinson and Velani [10]. We refer to [2, 6, 7] for refined modern results in this direction. For obvious reasons  $\psi$  is often referred to as an *approximating function*.

If one restricts the set of real numbers by putting restrictions on the set of integer vectors in which (1) is required to have infinitely many solutions, the problem is less studied. Constraints on the ‘denominator’ terms  $\mathbf{q}$  alone are relatively easily dealt with. For instance, appealing to a result of Schmidt [19], one can calculate the Lebesgue measure of such a set when coordinates of  $\mathbf{q}$  are restricted to values of prescribed polynomials. Furthermore, appealing to a result of Rynne [18], one may immediately determine the Hausdorff dimension of the set in question for more general restrictions. However, these results requires the ‘numerator’ term  $p$  to be free of restrictions.

Introducing constraints on both the denominator terms and the numerator term is a difficult problem, as many of the known general methods break down. In a series of papers, Harman [14, Chapter 6 and the references therein] dealt in detail with the case  $n = 1$  by studying classical Diophantine approximation of a real number by rationals for which the numerators and denominators come from specified sets. For  $n > 1$ , a linear condition on the numerator term  $p$  is easily dealt with by a change of variables. The case when the numerator term is restricted to being equal to zero is also well studied [9, 12, 15, 16]. However, to our knowledge the only complete metrical result with non-linear constraints on both numerator and denominator terms known to date is that of [3], in which the numerators and denominators are assumed to all be perfect squares. In this paper, we

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deal with a more general setup, in which these terms are required to be fixed, possibly unlike, powers of integers.

Finding the measure of such sets is not just an exercise in number theory. Indeed, Diophantine inequalities sometimes occur when studying the solutions to certain PDEs. It can often be shown that the exceptional set of points where these inequalities fail to hold is small, typically of zero Lebesgue measure. When ignored, the differential equations under consideration are guaranteed to have solutions and so a more acute understanding of the ‘size’ of these exceptional sets becomes a question of real interest.

For example, let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be periodic in each of its variables  $x_1, x_2$  and  $t$ , with periods  $\alpha, \beta$  and  $\gamma$  respectively. Assume also that  $f$  is a smooth function of each variable. The inhomogeneous wave equation studied in [3] is given by the PDE

$$\frac{\partial^2 u(\mathbf{x}, t)}{\partial t^2} - \frac{\partial^2 u(\mathbf{x}, t)}{\partial x_1^2} - \frac{\partial^2 u(\mathbf{x}, t)}{\partial x_2^2} = f(\mathbf{x}, t), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, t \in \mathbb{R}, \quad (2)$$

where  $u$  is a smooth, periodic solution with the same periods as  $f$ .

It is well known that the smoothness conditions on  $f$  are equivalent to the property that it has a Fourier series expansion of the form

$$f(\mathbf{x}, t) = \sum_{(a,b,c) \in \mathbb{Z}^3} f_{a,b,c} \exp \left( 2\pi i \left[ \frac{a}{\alpha} x_1 + \frac{b}{\beta} x_2 + \frac{c}{\gamma} t \right] \right),$$

in which the coefficients  $f_{a,b,c}$  decay suitably quickly. Any smooth solution  $u$  to (2) must satisfy a similar Fourier expansion. Upon comparing coefficients one can deduce that a sufficient condition for  $u$  to be smooth is that there is no real number  $\tau > 1$  such that the Diophantine inequality

$$\left| a^2 \frac{\gamma^2}{\alpha^2} + b^2 \frac{\gamma^2}{\beta^2} - c^2 \right| < \max \{ |a|, |b| \}^{-\tau} \quad (3)$$

holds for infinitely many  $(a, b, c) \in \mathbb{Z}^3$  with  $(a, b) \neq (0, 0)$ . In other words, a solution to (2) is guaranteed to exist if the ratio of certain functions of the periods of the functions in the PDE are not a set given by (1) under the condition that  $n = 2$  and the condition that  $q_1, q_2$  and  $p$  are all required to be perfect squares.

Of course, one could consider other PDEs, where the wave operator on the right hand side of (2) is replaced by a differential operator of the form

$$\frac{\partial^p}{\partial t^2} - \frac{\partial^n}{\partial x_1^2} - \frac{\partial^m}{\partial x_2^2}.$$

This would lead to a Diophantine obstruction, where the powers of 2 in (3) are replaced by  $n, m$  and  $p$ . It is therefore natural to investigate more general inequalities of the form (3) from a metrical point of view. For any triple  $(n, m, p) \in \mathbb{N}^3$  and any approximating function  $\psi$  define  $W_{n,m}^p(\psi)$  to be the set of vectors  $\mathbf{x} = (x_1, x_2) \in [0, 1]^2$  for which the inequality

$$|a^n x_1 + b^m x_2 - c^p| < \psi(h_{a,b})$$

holds for infinitely many  $(a, b, c) \in \mathbb{N}^2 \times \mathbb{Z}_{\geq 0}$ . Here, we have assigned a natural height  $h_{a,b} := \max(a^n, b^m)$  to each pair  $(a, b)$  of positive integers. We provide a Groshev-like criterion for the size of this set in terms of the convergence and divergence of a certain sum. In doing so, we generalise the result found in [3] corresponding to the case  $n = m = p = 2$ . In particular, we prove the following result, in which  $|\cdot|$  denotes normalized Lebesgue measure on  $[0, 1]^2$ .

**Theorem 1.** *Assume that either  $n = m = 1$  or  $\gcd(n, m) \geq 2$ . Then, for every approximating function  $\psi$  we have that*

$$|W_{n,m}^p(\psi)| = \begin{cases} 0, & \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} < \infty. \\ 1, & \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty. \end{cases}$$

When  $n = m = p = 1$  our result coincides with a case of the famous theorem of Groshev [13] and when  $n = m = p = 2$  the main result of [3]. Note that for some choices of natural numbers  $n, m$  and  $p$  the set  $W_{n,m}^p(\psi)$  will always be a Lebesgue null set (see §2.2.1).

In its own right, Theorem 1 gives no further information on how to distinguish between sets determined to have Lebesgue measure zero. Intuitively, the size of  $W_{n,m}^p(\psi)$  should still decrease as the rate of approximation governed by the approximating function  $\psi$  increases (assuming the sets concerned are non-empty). Hausdorff measure and dimension are the appropriate tools to distinguish amongst such exceptional sets. To this end, we now provide a criterion for the size of the set  $W_{n,m}^p(\psi)$  in terms of these tools. Throughout, by a *dimension function* we mean an increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(r) \rightarrow 0$  as  $r \rightarrow 0$ . As usual, we denote  $f$ -dimensional Hausdorff measure by  $\mathcal{H}^f$  and Hausdorff dimension by  $\dim_H$ . For precise definitions see §3.

**Theorem 2.** *Let  $\psi$  be an approximating function and assume that either  $n = m = p = 1$  or  $\gcd(n, m) \geq 2$ . Let  $f$  be a dimension function such that  $r^{-2}f(r)$  is monotonic and for notational convenience let  $g : r \rightarrow r^{-1}f(r)$  be another dimension function. Then,*

$$\mathcal{H}^f(W_{n,m}^p(\psi)) = \begin{cases} 0, & \sum_{(a,b) \in \mathbb{N}^2} g\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right) h_{a,b}^{1/p} < \infty. \\ \mathcal{H}^f([0, 1)^2), & \sum_{(a,b) \in \mathbb{N}^2} g\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right) h_{a,b}^{1/p} = \infty. \end{cases}$$

Note the added technical restriction on the range of  $n, m$  and  $p$  in comparison with Theorem 1. We will see in §2.2.1 that these assumptions are completely natural. Armed with this theorem, we are able to extend a result of Dodson [11] upon setting  $f : r \rightarrow r^s$  for some  $s > 0$ .

**Corollary 1.** *Let  $\psi$  be an approximating function and assume that either  $n = m = p = 1$  or  $\gcd(n, m) \geq 2$ . Define the quantity  $\lambda_\psi \in [0, \infty]$  by*

$$\lambda_\psi := \liminf_{r \rightarrow \infty} \frac{-\log \psi(2^r)}{r \log 2}.$$

*If  $\lambda_\psi < \infty$ , then*

$$\dim_H(W_{n,m}^p(\psi)) = 1 + \min \left\{ 1, \frac{\frac{1}{n} + \frac{1}{m} + \frac{1}{p}}{\lambda_\psi + 1} \right\}.$$

To be precise, the result [11] corresponds to the case  $n = m = p = 1$  in the above statement, whereas if  $n = m = p = 2$  we have reproduced Corollary 3.3 from [3].

When  $\psi : r \rightarrow r^{-\tau}$  for some  $\tau > 1$ , the following Hausdorff dimension statement can readily be obtained. This is in perfect analogue with Corollary 3.4 from [3].

**Corollary 2.** *Assume that either  $n = m = p = 1$  or  $\gcd(n, m) \geq 2$ . Let  $\tau > 1$ , then*

$$\dim_H (W_{n,m}^p (\psi : r \rightarrow r^{-\tau})) = 1 + \min \left\{ 1, \frac{\frac{1}{n} + \frac{1}{m} + \frac{1}{p}}{\tau + 1} \right\}.$$

In fact, Theorem 2 reveals a lot more than just the Hausdorff dimension of the sets concerned. It also implies for example that  $\mathcal{H}^s (W_{n,m}^p (\psi : r \rightarrow r^{-\tau})) = \infty$  at the critical exponent  $s = \dim_H (W_{n,m}^p (\psi : r \rightarrow r^{-\tau}))$ .

## 2. PROOF OF THEOREM 1

We will use the following notation throughout the proof. Fix any two natural numbers  $a$  and  $b$ . Then, for every  $c \in \mathbb{Z}_{\geq 0}$  let

$$\ell_{a,b}(c) := \{(x, y) \in [0, 1)^2 : |a^n x + b^m y - c^p| < \psi(h_{a,b})\}$$

and in turn let

$$\ell_{a,b} = \bigcup_{c \in \mathbb{Z}_{\geq 0}} \ell_{a,b}(c).$$

Each set  $\ell_{a,b}(c)$  is simply a ‘strip’ in  $[0, 1)^2$  consisting of a segment of a certain neighbourhood of the line  $y = (-a^n/b^m)x + c^p/b^m$ . The set  $\ell_{a,b}$  is the disjoint union as  $c$  runs over  $\mathbb{Z}_{\geq 0}$  of all such strips which are non-empty. Moreover, this notation gives us a very convenient way of expressing the set  $W_{n,m}^p(\psi)$ . Indeed, we have

$$W_{n,m}^p(\psi) = \{(x, y) \in [0, 1)^2 : (x, y) \in \ell_{a,b} \text{ for infinitely many pairs } (a, b) \in \mathbb{N}^2\}.$$

As is now commonplace in number theory we will often appeal to Vinogradov notation rather than ‘big O’ notation to allow for neatness of exposition. For the unfamiliar reader we mean by  $f \ll g$  that  $f(t) = \mathcal{O}(g(t))$  as  $t \rightarrow \infty$  and by  $f \asymp g$  that both  $f \ll g$  and  $f \gg g$ . We consider  $n, m$  and  $p$  to be fixed throughout the proof and any implied constants may depend on these integers alone.

We begin the proof of Theorem 1 by dealing with the case when the volume sum converges. As is usual for results of this type, it takes the form of a simple covering argument.

**2.1. Proof of the convergence part.** For any fixed  $c$  it is easily verified that the strip  $\ell_{a,b}(c)$  has measure at most  $2\sqrt{2}\psi(h_{a,b})/\sqrt{a^{2n} + b^{2m}} \ll \psi(h_{a,b})/h_{a,b}$  and a simple calculation yields that there are at most  $2h_{a,b}^{1/p} + 1$  non-empty strips in the union  $\ell_{a,b}$ . Hence,

$$|\ell_{a,b}| \ll \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}}.$$

Furthermore, the function  $h_{a,b} : \mathbb{N}^2 \rightarrow \mathbb{N}$  only takes values which are  $n$ -th or  $m$ -th powers, and so for any natural number  $h$  not of this form we have

$$\bigcup_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b}=h}} \ell_{a,b} = \emptyset.$$

Therefore,

$$\begin{aligned}
\sum_{h=1}^{\infty} \left| \bigcup_{(a,b) \in \mathbb{N}^2, h_{a,b}=h} \ell_{a,b} \right| &= \sum_{g_1=1}^{\infty} \left| \bigcup_{b^m \leq g_1^n} \ell_{g_1,b} \right| + \sum_{g_2=1}^{\infty} \left| \bigcup_{a^n < g_2^m} \ell_{a,g_2} \right| \\
&\ll \sum_{g_1=1}^{\infty} \sum_{\substack{b \in \mathbb{N} \\ b^m \leq g_1^n}} \frac{\psi(g_1^n)}{g_1^{n(1-1/p)}} + \sum_{g_2=1}^{\infty} \sum_{\substack{a \in \mathbb{N} \\ a^n < g_2^m}} \frac{\psi(g_2^m)}{g_2^{m(1-1/p)}} \\
&= \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b}=a^n}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} + \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b}=b^m > a^n}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} \\
&= \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} < \infty.
\end{aligned}$$

Since

$$W_{n,m}^p(\psi) = \bigcap_{g=1}^{\infty} \bigcup_{h=g}^{\infty} \left( \bigcup_{(a,b) \in \mathbb{N}^2, h_{a,b}=h} \ell_{a,b} \right),$$

it follows from the ‘convergence part’ of the famous Borel-Cantelli lemma in probability theory that  $W_{n,m}^p(\psi)$  is a null set as required.

**2.2. Preliminaries for the proof of the divergence part.** Proving the validity of Theorem 1 for the case when the volume sum

$$\sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} \quad (4)$$

diverges constitutes the main difficulty in the proof of Theorem 1 as a whole and will require some very delicate calculations. As such, before we proceed we are first required to provide some auxiliary lemmas and outline some important general observations.

**2.2.1. An important observation.** Since  $\psi(r) \rightarrow 0$  as  $r \rightarrow \infty$  we may assume that  $\psi(r) \leq 1$  for  $r$  sufficiently large. Consider the cases when we do not have  $n = m = 1$ . For these cases, observe that since  $\gcd(n, m) \geq 2$  we may assume throughout that either  $n = m = 2$  or

$$p \leq \frac{nm}{nm - n - m}. \quad (5)$$

For, if the latter were not the case it would follow that

$$\min \{n(1 - 1/p - 1/m), m(1 - 1/p - 1/n)\} > 1,$$

and so for any sufficiently large  $H \in \mathbb{N}$  we would have

$$\begin{aligned}
\sum_{\substack{(a,b) \in \mathbb{N}^2: \\ h_{a,b} \geq H}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} &= \sum_{g_1 \geq \lfloor H^{1/n} \rfloor} \sum_{\substack{b \in \mathbb{N} \\ b^m \leq g_1^n}} \frac{\psi(g_1^n)}{g_1^{n(1-1/p)}} + \sum_{g_2 \geq \lfloor H^{1/m} \rfloor} \sum_{\substack{a \in \mathbb{N} \\ a^n < g_2^m}} \frac{\psi(g_2^m)}{g_2^{m(1-1/p)}} \\
&\ll \sum_{g_1 \geq \lfloor H^{1/n} \rfloor} \frac{g_1^{n/m}}{g_1^{n(1-1/p)}} + \sum_{g_2 \geq \lfloor H^{1/m} \rfloor} \frac{g_2^{m/n}}{g_2^{m(1-1/p)}} < \infty,
\end{aligned}$$

a contradiction. Moreover, in view of §2.1 this observation implies that the set  $W_{n,m}^p(\psi)$  has measure zero for every approximating function  $\psi$  whenever (5) does not hold (assuming we don't have  $n = m = 2$ ). Similarly, when  $n = m = 1$  one can easily verify that either  $p = 1$  or the set  $W_{1,1}^p(\psi)$  always has measure zero.

**2.2.2. Restrictions on the integers  $(a, b)$ .** It will be imperative to our proof that we exclude a certain class of integer pairs  $(a, b) \in \mathbb{N}^2$  from our calculations. Firstly, we will exclude those pairs for which  $a$  and  $b$  are not coprime. The reason for this is that it will guarantee that the strip  $\ell_{a,b}$  is not parallel to any other strip we might consider. In view of the method outlined in §2.2.3 this will be very important, as otherwise the intersection of any two such strips may be very large.

Secondly, we will assume that the resonant lines at the centre of all our strips lie in some 'cone'; that is, the angle of incline of each strip  $\ell_{a,b}$  is neither too steep or too flat. The reason for this assumption will become apparent as our proof progresses. To be precise, for the rest of the proof we will work exclusively with pairs  $(a, b) \in \mathcal{N}_{n,m}$ , where  $\mathcal{N}_{n,m} \subset \mathbb{N}^2$  denotes the set of pairs  $(a, b) \in \mathbb{N}^2$  satisfying the conditions

$$\gcd(a, b) = 1, \quad \frac{1}{2} < \frac{a^n}{b^m} < 2. \quad (6)$$

Before we proceed we must first ensure that this thinning out of the sequence of sets  $\ell_{a,b}$  does not effect the implication of our proof. To see that it does not, notice that the set

$$V_{n,m}^p(\psi) = \{(x, y) \in [0, 1)^2 : (x, y) \in \ell_{a,b} \text{ for infinitely many pairs } (a, b) \in \mathcal{N}_{n,m}\}$$

is a subset of  $W_{n,m}^p(\psi)$ . Therefore, if we can prove that  $V_{n,m}^p(\psi)$  has full measure then it will readily follow that  $W_{n,m}^p(\psi)$  also enjoys this property. Our proof of Theorem 1 would then be complete modulo the following proposition, which demonstrates that the sequence of strips  $\ell_{a,b}$  for  $(a, b) \in \mathcal{N}_{n,m}$  is 'rich' enough to entirely determine whether the volume sum (4) diverges.

**Proposition 1.** *For any approximating function  $\psi$  and any triple  $(n, m, p) \in \mathbb{N}^3$  we have*

$$\sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty \quad \Longleftrightarrow \quad \sum_{(a,b) \in \mathcal{N}_{n,m}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty.$$

To prove this proposition we will require the following two lemmas. Throughout  $\varphi$  will denote Euler's totient function and  $\sigma$  will denote the divisor function.

**Lemma 1.** *Choose a fixed natural number  $t$ , and then for any other natural number  $Q$  denote by  $\gamma_t(Q)$  the cardinality of the set  $\{q \leq Q : \gcd(t, q) = 1\}$ . Then,*

$$\gamma_t(Q) = \frac{\varphi(t)}{t}Q + \epsilon_t(Q),$$

where  $\epsilon_t(Q) : \mathbb{N} \rightarrow \mathbb{R}$  is an error function satisfying  $|\epsilon_t(Q)| \leq \sigma(t)$  for every  $Q \in \mathbb{N}$ .

*Proof.* Recall the basic property of the Möbius function  $\mu$  that for  $t \in \mathbb{N}$  we have

$$\sum_{d|t} \mu(d) = \begin{cases} 1 & \text{if } t = 1. \\ 0 & \text{if } t > 1. \end{cases}$$

With this and other standard number theoretical identities in mind, for any  $Q \in \mathbb{N}$  we have

$$\begin{aligned}\gamma_t(Q) &= \sum_{q=1}^Q \sum_{d \mid \gcd(t,q)} \mu(d) = \sum_{d \mid t} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor \\ &= Q \sum_{d \mid t} \frac{\mu(d)}{d} - \sum_{d \mid t} \mu(d) \left\{ \frac{Q}{d} \right\} \\ &= \frac{\varphi(t)}{t} Q - \sum_{d \mid t} \mu(d) \left\{ \frac{Q}{d} \right\},\end{aligned}$$

as required. □

**Lemma 2.** *For any fixed real number  $z > 0$ , we have*

$$\sum_{q=1}^Q q^{z-1} \varphi(q) = \frac{6}{\pi^2(z+1)} Q^{z+1} + \mathcal{O}(Q^z)$$

as  $Q \rightarrow \infty$ .

*Proof.*

$$\begin{aligned}\sum_{q=1}^Q q^{z-1} \varphi(q) &= \sum_{q=1}^Q q^z \sum_{d \mid q} \frac{\mu(d)}{d} = \sum_{q=1}^Q \sum_{dd'=q} \mu(d) d^{z-1} (d')^z \\ &= \sum_{dd' \leq Q} \mu(d) d^{z-1} (d')^z \\ &= \sum_{d=1}^Q \mu(d) d^{z-1} \sum_{d'=1}^{\lfloor \frac{Q}{d} \rfloor} (d')^z.\end{aligned}$$

By the famous Taylor-Maclaurin summation formula and the binomial theorem respectively we have for the inner sum above that

$$(z+1) \sum_{d'=1}^{\lfloor \frac{Q}{d} \rfloor} (d')^z = \left\lfloor \frac{Q}{d} \right\rfloor^{z+1} + \mathcal{O}\left(\left\lfloor \frac{Q}{d} \right\rfloor^z\right) = \left(\frac{Q}{d}\right)^{z+1} + \mathcal{O}\left(\left(\frac{Q}{d}\right)^z\right)$$

as  $Q \rightarrow \infty$ . Therefore,

$$\begin{aligned}\sum_{q=1}^Q q^{z-1} \varphi(q) &= \frac{1}{z+1} \sum_{d=1}^Q \mu(d) d^{z-1} \left( \left(\frac{Q}{d}\right)^{z+1} + \mathcal{O}\left(\left(\frac{Q}{d}\right)^z\right) \right) \\ &= \frac{Q^{z+1}}{z+1} \sum_{d=1}^Q \frac{\mu(d)}{d^2} + \mathcal{O}\left(Q^z \sum_{d=1}^Q \frac{\mu(d)}{d}\right)\end{aligned}$$

as  $Q \rightarrow \infty$ . It is well known (e.g. [1] Theorems 3.13 & 3.2(c)) that for any  $Q \in \mathbb{N}$  we have

$$\left| \sum_{d=1}^Q \frac{\mu(d)}{d} \right| \leq 1 \quad \text{and} \quad \sum_{d=Q+1}^{\infty} \frac{1}{d^2} \ll \frac{1}{Q},$$

and also (e.g. [1] Theorem 11.5) that

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)},$$

where  $\zeta$  is the Riemann zeta function. Hence,

$$\begin{aligned} \sum_{q=1}^Q q^{z-1} \varphi(q) &= \frac{Q^{z+1}}{z+1} \sum_{d=1}^Q \frac{\mu(d)}{d^2} + \mathcal{O}(Q^t) \\ &= \frac{Q^{z+1}}{z+1} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \frac{Q^{z+1}}{z+1} \sum_{d=Q+1}^{\infty} \frac{\mu(d)}{d^2} + \mathcal{O}(Q^z) \\ &= \frac{1}{(z+1)\zeta(2)} Q^{z+1} + \mathcal{O}(Q^z) \end{aligned}$$

as  $Q \rightarrow \infty$ . □

We may now prove Proposition 1. It suffices to show that

$$\sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty \quad \implies \quad \sum_{(a,b) \in \mathcal{N}_{n,m}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty$$

since the complementary implication is obvious. To do this we will show that if the latter sum converges then so does the former.

In order to proceed we first partition the set  $\mathcal{N}_{n,m}$  by separating the heights  $h_{a,b}$  into diadic blocks. For any  $t \in \mathbb{Z}_{\geq 0}$  let

$$A_t := \{(a, b) \in \mathcal{N}_{n,m} : a^n > b^m, 2^{t+1} > a \geq 2^t\},$$

and let

$$B_t := \{(a, b) \in \mathcal{N}_{n,m} : b^m > a^n, 2^{t+1} > b \geq 2^t\}.$$

Note that by definition  $\mathcal{N}_{n,m}$  is precisely the disjoint union  $\cup_{t \in \mathbb{Z}_{\geq 0}} (A_t \cup B_t)$ . We will denote by  $\alpha_t$  the cardinality of the set  $A_t$  and by  $\beta_t$  the cardinality of the set  $B_t$ . We may re-express the set  $A_t$  in the following way:

$$A_t = \left\{ (a, b) \in \mathbb{N}^2 : \gcd(a, b) = 1, \left\lfloor a^{\frac{n}{m}} \right\rfloor > b > \left\lfloor 2^{\frac{1}{m}} a^{\frac{n}{m}} \right\rfloor, 2^{t+1} > a \geq 2^t \right\}.$$

Recall the following well known expression (e.g. [1] Theorem 3.3) describing the behaviour of the divisor summatory function:

$$\sum_{q=1}^Q \sigma(q) = Q \log Q + \mathcal{O}(Q)$$

as  $Q \rightarrow \infty$ . With reference to the notation of Lemma 1, an immediate consequence of the above statement is that for any sequence of natural numbers  $\{N_q\}_{q=1}^{\infty}$  we have

$$\left| \sum_{q=2^t+1}^{2^{t+1}} \epsilon_q(N_q) \right| \leq \sum_{q=2^t+1}^{2^{t+1}} |\epsilon_q(N_q)| \leq \sum_{q=2^t+1}^{2^{t+1}} \sigma(q) \ll t 2^t. \quad (7)$$

Finally, we recall (e.g. [1] Chapter 3, Ex. 5(b)) the property that

$$\sum_{d=1}^Q \frac{\varphi(d)}{d} = \frac{Q}{\zeta(2)} + \mathcal{O}(\log Q),$$



as  $Q \rightarrow \infty$ . Hence, on applying Lemma 1 to the set  $A_t$  and utilising (7) we conclude that

$$\begin{aligned}
\alpha_t &= \sum_{a=2^t+1}^{2^{t+1}} \left( \gamma_a \left( \left\lfloor a^{\frac{n}{m}} \right\rfloor - 1 \right) - \gamma_a \left( \left\lfloor 2^{\frac{1}{m}} a^{\frac{n}{m}} \right\rfloor \right) \right) \\
&= \sum_{a=2^t+1}^{2^{t+1}} \left( \frac{\varphi(a)}{a} \left( \left\lfloor a^{\frac{n}{m}} \right\rfloor - 1 - \left\lfloor 2^{\frac{1}{m}} a^{\frac{n}{m}} \right\rfloor \right) + \epsilon_a \left( \left\lfloor a^{\frac{n}{m}} \right\rfloor - 1 \right) - \epsilon_a \left( \left\lfloor 2^{\frac{1}{m}} a^{\frac{n}{m}} \right\rfloor \right) \right) \\
&= \left( 1 - 2^{\frac{1}{m}} \right) \sum_{a=2^t+1}^{2^{t+1}} a^{\frac{n}{m}-1} \varphi(a) + \mathcal{O}(t2^t)
\end{aligned}$$

as  $t \rightarrow \infty$ . By Lemma 2 it follows that

$$\begin{aligned}
\alpha_t &= \left( 1 - 2^{\frac{1}{m}} \right) \frac{2^{(\frac{n}{m}+1)(t+1)} - 2^{(\frac{n}{m}+1)t}}{\left( \frac{n}{m} + 1 \right) \zeta(2)} + \mathcal{O} \left( \max \left( t2^t, 2^{\frac{n}{m}t} \right) \right) \\
&= \frac{\left( 1 - 2^{\frac{1}{m}} \right) \left( 2^{(\frac{n}{m}+1)} - 1 \right)}{\left( \frac{n}{m} + 1 \right) \zeta(2)} 2^{(\frac{n}{m}+1)t} + \mathcal{O} \left( \max \left( t2^t, 2^{\frac{n}{m}t} \right) \right)
\end{aligned}$$

as  $t \rightarrow \infty$ , and so  $\alpha_t \asymp 2^{(\frac{n}{m}+1)t}$ . Analogously, one can show a similar result concerning the set  $B_t$ ; that is, for any  $t \in \mathbb{Z}_{\geq 0}$  we have  $\beta_t \asymp 2^{(\frac{m}{n}+1)t}$ .

To complete the proof, we deduce that

$$\begin{aligned}
\sum_{(a,b) \in \mathcal{N}_{n,m}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} &\asymp \sum_{\substack{(a,b) \in \mathcal{N}_{n,m} \\ h_{a,b} = a^n}} \frac{\psi(a^n)}{a^{n(1-1/p)}} + \sum_{\substack{(a,b) \in \mathcal{N}_{n,m} \\ h_{a,b} = b^m}} \frac{\psi(b^m)}{b^{m(1-1/p)}} \\
&= \sum_{t=0}^{\infty} \sum_{(a,b) \in A_t} \frac{\psi(a^n)}{a^{n(1-1/p)}} + \sum_{s=0}^{\infty} \sum_{(a,b) \in B_s} \frac{\psi(b^m)}{b^{m(1-1/p)}} \\
&\gg \sum_{t=0}^{\infty} \frac{\psi(2^{(t+1)n})}{2^{(t+1)n(1-1/p)}} \sum_{(a,b) \in A_t} 1 + \sum_{s=0}^{\infty} \frac{\psi(2^{(s+1)m})}{2^{(s+1)m(1-1/p)}} \sum_{(a,b) \in B_s} 1 \\
&= \sum_{t=0}^{\infty} \frac{\psi(2^{(t+1)n})}{2^{(t+1)n(1-1/p)}} \alpha_t + \sum_{s=0}^{\infty} \frac{\psi(2^{(s+1)m})}{2^{(s+1)m(1-1/p)}} \beta_s \\
&\asymp \sum_{t=0}^{\infty} \frac{2^{t+1} \psi(2^{(t+1)n})}{2^{(t+1)n(1-\frac{1}{p}-\frac{1}{m})}} + \sum_{s=0}^{\infty} \frac{2^{s+1} \psi(2^{(s+1)m})}{2^{(s+1)m(1-\frac{1}{p}-\frac{1}{n})}} \\
&\asymp \sum_{a=1}^{\infty} \frac{\psi(a^n)}{a^{n(1-\frac{1}{p}-\frac{1}{m})}} + \sum_{b=1}^{\infty} \frac{\psi(b^m)}{b^{m(1-\frac{1}{p}-\frac{1}{n})}} \\
&\asymp \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b} = a^n}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} + \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b} = b^m}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} \\
&= \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}}.
\end{aligned}$$

Thus, if the first sum converges then so does the last and the proposition is proven by a contrapositive argument.

2.2.3. *Some auxiliary lemmata and the general strategy.* For the most part our method for proving the bulk of Theorem 1 will follow the general strategy outlined in [3]. Indeed, the basis of our proof will be the following consequence of Lebesgue's density theorem.

**Lemma 3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^t$  and let  $E$  be a Borel subset of  $\mathbb{R}^t$ . If there exist strictly positive constant  $r_0$  such that for any ball  $B$  in  $\Omega$  of radius  $r(B) < r_0$  we have*

$$|E \cap B| \gg |B|, \quad (8)$$

*where the implied constant is independent of  $B$ , then  $E$  has full measure in  $\Omega$ .*

Remark: We have previously insisted that all implied constants in the Vinogradov symbols may depend only on  $n, m$  and  $p$ , so the condition above of independence from  $B$  may seem obsolete here. However, we include it in our statement as the reader may find other uses for this variant of the Lebesgue density theorem.

For technical reasons, we will take  $\Omega$  to be the set  $[\epsilon, 1]^2$  for some arbitrarily small  $\epsilon > 0$ . Then upon setting  $E = V_{n,m}^p(\psi)$  and  $r_0 = \epsilon$ , where as before

$$V_{n,m}^p(\psi) = \{(x, y) \in [0, 1]^2 : (x, y) \in \ell_{a,b} \text{ for infinitely many pairs } (a, b) \in \mathcal{N}_{n,m}\},$$

it follows subject to proving (8) that the set  $V_{n,m}^p(\psi) \cap [\epsilon, 1]^2$  has measure  $(1 - \epsilon)^2$ . A proof of Theorem 1 will follow upon letting  $\epsilon \rightarrow 0$ .

The key to establishing (8) with  $E = V_{n,m}^p(\psi)$  and  $r_0 = \epsilon$  will be the following lemma.

**Lemma 4.** *Let  $E_t$  be a sequence of measurable sets which are **quasi-independent on average**; that is, the sequence  $E_t$  satisfies*

$$\sum_{t=1}^{\infty} |E_t| = \infty \quad (9)$$

*and there exists some strictly positive constant  $\alpha$  for which*

$$\sum_{s,t=1}^Q |E_s \cap E_t| \leq \frac{1}{\alpha} \left( \sum_{t=1}^Q |E_t| \right)^2$$

*for all  $Q \geq 1$ . Then,*

$$\left| \limsup_{t \rightarrow \infty} E_t \right| \geq \alpha$$

*Proof.* The lemma follows immediately from a generalisation of the ‘divergent part’ of the Borel-Cantelli lemma (e.g., [20] Lemma 5), which states that for any sequence of measurable sets  $E_t$  satisfying (9) we have

$$\left| \limsup_{t \rightarrow \infty} E_t \right| \geq \limsup_{Q \rightarrow \infty} \frac{\left( \sum_{t=1}^Q |E_t| \right)^2}{\sum_{s,t=1}^Q |E_s \cap E_t|}.$$

□

The remainder of this section will be dedicated to proving the following proposition.

**Proposition 2.** *For any ball  $B \in [\epsilon, 1]^2$  of radius  $r < \epsilon$ , the sequence of sets  $\{\ell_{a,b} \cap B\}_{(a,b) \in \mathcal{N}_{n,m}}$  is quasi-independent on average. In particular,*

$$\sum_{(a,b) \in \mathcal{N}_{n,m}} |\ell_{a,b} \cap B| = \infty \quad (10)$$

and

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ h_1 \leq H, h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll \frac{1}{|B|} \left( \sum_{(a, b) \in \mathcal{N}_{n, m}, h_{a, b} \leq H} |\ell_{a, b} \cap B| \right)^2, \quad (11)$$

for all  $H \geq 1$ .

By construction we have

$$\limsup_{h_{a, b} \rightarrow \infty} (\ell_{a, b} \cap B) = V_{n, m}^p(\psi) \cap B.$$

In view of Lemma 4 and the above discussion, Proposition 2 establishes that (8) holds with  $E = V_{n, m}^p(\psi)$  and  $r_0 = \epsilon$ , and in turn Theorem 1.

**2.3. Estimating the measure of  $\ell_{a, b} \cap B$ .** In order to prove Proposition 2 we wish to estimate the measure of the intersections  $\ell_{a, b} \cap B$  for  $(a, b) \in \mathcal{N}_{n, m}$ . To do this, for each  $(a, b) \in \mathcal{N}_{n, m}$  we first require to estimate the number of integers  $c$  for which  $\ell_{a, b}(c) \cap B \neq \emptyset$ .

Fix the two integers  $a$  and  $b$  and set  $h = h_{a, b}$ . If  $\ell_{a, b}(c) \cap B \neq \emptyset$  then there exists  $(x, y) \in B$  such that  $|a^n x + b^m y - c| < \psi(h)$ . If  $h_{a, b}$  is sufficiently large then we may assume  $\psi(h) < \epsilon$  and so

$$c^p < a^n x + b^m y + \psi(h) < a^n + b^m + \epsilon < 2h + 1.$$

Consequently, we must have that  $c < 3h^{1/p}$ . On the other hand,

$$c^p > a^n x + b^m y - \psi(h) > \epsilon(a^n + b^m) - \epsilon > \epsilon(h - 1),$$

and so

$$\frac{\epsilon^{1/p}}{2} h^{1/p} < c < 3h^{1/p}. \quad (12)$$

For ease of notation, for each natural number  $c$  let

$$R_{a, b}(c) := \{(x, y) \in [\epsilon, 1]^2 : a^n x + b^m y - c^p = 0\}$$

denote the intersection of the line  $a^n x + b^m y - c^p = 0$  in  $\mathbb{R}^2$  with  $[\epsilon, 1]^2$ . Then,  $\ell_{a, b}(c) \cap B \neq \emptyset$  if and only if one of the following situations arises:

1.  $R_{a, b}(c)$  does not intersect  $B$  but passes within a certain small neighbourhood of it. To be precise, the shortest distance from the line  $R_{a, b}(c)$  to the centre  $(x_0, y_0)$  must not exceed  $2\psi(h)/\sqrt{a^{2n} + b^{2m}} + r$ .
2.  $R_{a, b}(c) \cap B \neq \emptyset$ .

It is clear that there are at a most two distinct values of  $c$  for which the former case is satisfied, and to estimate how many the latter contributes we proceed as follows. Denote by  $(x_0, y_0)$  the centre of the ball  $B$ . Then  $R_{a, b}(c) \cap B \neq \emptyset$  if and only if there exists  $(x, y) \in R_{a, b}(c)$  which can be written in the form

$$x = x_0 + tr \cos \theta, \quad y = y_0 + tr \sin \theta, \quad \text{for some } t \in [0, 1) \text{ and } \theta \in [0, 2\pi).$$

This holds if and only if

$$a^n x_0 + b^m y_0 - r\sqrt{a^{2n} + b^{2m}} < c^p < a^n x_0 + b^m y_0 + r\sqrt{a^{2n} + b^{2m}}.$$

Since we have assumed the radius  $r < \epsilon$  and that  $x_0, y_0 \geq \epsilon$ , the quantity on the left hand side above is strictly positive. Therefore, all possible choices for the integer  $c$  for which  $R_{a,b}(c) \cap B \neq \emptyset$  lie in an interval of length  $n_{a,b,B}$  satisfying

$$\begin{aligned} n_{a,b,B} &= r(a^{2n} + b^{2m})^{\frac{1}{2p}} \left( \frac{1}{r} \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} + r \right)^{\frac{1}{p}} - \frac{1}{r} \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} - r \right)^{\frac{1}{p}} \right) \\ &= r(a^{2n} + b^{2m})^{\frac{1}{2p}} \left( \frac{\left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} + r \right)^{\frac{2}{p}} - \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} - r \right)^{\frac{2}{p}}}{\left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} + r \right)^{\frac{1}{p}} + \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} - r \right)^{\frac{1}{p}}} \right) \frac{1}{r}. \end{aligned} \quad (13)$$

If  $p = 1$  then  $n_{a,b,B} = 2r(a^{2n} + b^{2m})^{\frac{1}{2}}$ . Otherwise, by utilising the assumptions that  $0 < r < \epsilon$  and  $\epsilon \leq x_0, y_0 \leq 1$  and the trivial inequality

$$\frac{a^n + b^m}{2} \leq \sqrt{a^{2n} + b^{2m}} \leq a^n + b^m,$$

the denominator of the bracketed term in (13) is easily seen to satisfy

$$\begin{aligned} \epsilon^{\frac{1}{p}} &< (\epsilon + r)^{\frac{1}{p}} + (\epsilon - r)^{\frac{1}{p}} \leq \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} + r \right)^{\frac{1}{p}} + \left( \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}} - r \right)^{\frac{1}{p}} \\ &\leq (2 + r)^{\frac{1}{p}} + (2 - r)^{\frac{1}{p}} \leq 4. \end{aligned}$$

When  $p = 2$  the numerator of the bracketed term in (13) is simply equal to  $2r$ . For  $p \geq 3$ , we may appeal to Newton's well known generalised binomial theorem for the case of non-integer powers; that is, if  $\alpha$  and  $\beta$  are real numbers with  $|\alpha| > |\beta|$  and  $\gamma$  is a real number (but not an integer) then

$$(\alpha + \beta)^\gamma = \sum_{t=0}^{\infty} \frac{(\alpha)_t}{t!} \alpha^{\gamma-t} \beta^t,$$

where  $(\gamma)_t = \gamma(\gamma-1)(\gamma-2) \cdots (\gamma-t+1)$  is the Pochhammer symbol denoting the falling factorial. For notational convenience let

$$z = z(a, b, n, m, p, x_0, y_0) := \frac{a^n x_0 + b^m y_0}{\sqrt{a^{2n} + b^{2m}}}.$$

It follows that

$$(z + r)^{\frac{2}{p}} - (z - r)^{\frac{2}{p}} = 2 \sum_{t=0}^{\infty} \frac{\left(\frac{2}{p}\right)_{2t+1}}{(2t+1)!} z^{\frac{2}{p}-2t-1} r^{2t+1}.$$

Note that  $(\gamma)_{2t+1}$  is positive for any  $\gamma \in (0, 1)$  and so every individual term in this sum is positive. The sum itself is therefore trivially bounded below by its initial term. For an upper bound we appeal to the well known fact that for any  $\gamma \in (0, 1)$  we have

$$\left| \frac{(\gamma)_t}{t!} \right| \leq \frac{e^{(\gamma+1)^2}}{(t+1)^{1+\gamma}}, \quad \text{for all } t \geq 1.$$

For completeness, we include the derivation of this inequality here. For  $t \geq 1$  it follows from the well known inequality of arithmetic and geometric means that

$$\begin{aligned} \left| \frac{(\gamma)_t}{t!} \right|^2 &= \prod_{s=1}^t \left| \frac{\gamma+1-s}{s} \right|^2 \leq \left( \frac{1}{t} \sum_{s=1}^t \left| \frac{\gamma+1-s}{s} \right|^2 \right)^t \\ &= \left( 1 + \frac{1}{t} \left( (\gamma+1)^2 \sum_{s=1}^t \frac{1}{s^2} - 2(\gamma+1) \sum_{s=1}^t \frac{1}{s} \right) \right)^t. \end{aligned}$$

Notice that for  $r \geq -t$ , we have  $(1 + r/t)^t \leq e^r$ . It is easy to check by hand for  $t = 1, 2, 3$  that

$$r_t := (\gamma + 1)^2 \sum_{s=1}^t \frac{1}{s^2} - 2(\gamma + 1) \sum_{s=1}^t \frac{1}{s} > -t.$$

For  $t \geq 3$  the elementary upper bound  $\sum_{s=1}^t s^{-1} \leq \log t + 1$  implies that

$$r_t > (\gamma + 1)(\gamma + 1 - 2(1 + \log t)) > -2(\log t - 1) > -t.$$

Furthermore, since  $\sum_{s=1}^t s^{-1} \geq \log(t + 1)$  we have

$$\left| \frac{(\gamma)_t}{t!} \right| \leq e^{r_t/2} \leq e^{(\gamma+1)^2 - (\gamma+1)\log(t+1)} = \frac{e^{(\gamma+1)^2}}{(t+1)^{1+\gamma}}$$

as required.

In view of the above we therefore have

$$\begin{aligned} \frac{2^{\frac{2}{p}} r}{p} &\leq \frac{2r}{p z^{1-\frac{2}{p}}} < \sum_{t=0}^{\infty} \frac{\left(\frac{2}{p}\right)_{2t+1}}{(2t+1)!} z^{\frac{2}{p}-2t-1} r^{2t+1} \\ &\leq \frac{e^{\left(\frac{2}{p}+1\right)^2} r}{(2z)^{1-\frac{2}{p}}} \sum_{t=0}^{\infty} \left(\frac{r}{z}\right)^{2t} \frac{1}{(t+1)^{1+\frac{2}{p}}} \\ &\leq \frac{e^{\left(\frac{2}{p}+1\right)^2} r}{(2\epsilon)^{1-\frac{2}{p}}} \sum_{t=0}^{\infty} \frac{1}{(t+1)^{1+\frac{2}{p}}} \\ &\leq \frac{Cr}{\epsilon^{1-\frac{2}{p}}}, \end{aligned}$$

where  $C$  is some positive real number depending only on  $p$ . It follows that for any  $p \geq 1$  we have

$$\frac{r}{2^{1-\frac{2}{p}}} (a^{2n} + b^{2m})^{\frac{1}{2p}} \leq n_{a,b,B} \leq \frac{2Cr}{\epsilon^{1-\frac{1}{p}}} (a^{2n} + b^{2m})^{\frac{1}{2p}}.$$

Combining the two cases (1. and 2.) outlined earlier in this subsection we conclude that the number of possible choices for  $c$  for which  $\ell_{a,b}(c) \cap B \neq \emptyset$  must be  $\asymp r h^{1/p}$ .

We may now estimate the measure of the intersection  $\ell_{a,b} \cap B$ . For each integer  $c$  we have the trivial upper bound

$$|\ell_{a,b}(c) \cap B| \leq \frac{4r\psi(h)}{\sqrt{a^{2n} + b^{2m}}} \ll \frac{r\psi(h)}{h}.$$

However, we have no general lower bound on the  $\ell_{a,b}(c) \cap B$  as the intersection may be even as small as a single point. To counter this problem we consider a subset of those integers  $c$  satisfying  $\ell_{a,b}(c) \cap B \neq \emptyset$  for which this intersection is sufficiently large.

Let  $\frac{1}{2}B$  be the ball  $B$  scaled by one half; that is,  $\frac{1}{2}B$  is the open ball in  $\Omega$  with centre  $(x_0, y_0)$  and radius  $r/2$ . It is easy to see that for  $h$  sufficiently large that if  $\ell_{a,b}(c)$  intersects  $\frac{1}{2}B$  then  $|\ell_{a,b}(c) \cap B| \geq 2r\psi(h)/\sqrt{a^{2n} + b^{2m}} \asymp r\psi(h)/h$ . As before, the number of possible choices of  $c$  for which  $\ell_{a,b}(c) \cap \frac{1}{2}B \neq \emptyset$  is  $\asymp r h^{1/p}$ .

One should note that the upper bound obtained for the number of choices of  $c$  in the calculations above coincides with the trivial one (that is, the diameter of the ball divided by the maximum distance between two adjacent lines as will be calculated in §2.4). However, it is the lower bound which is of importance to us in proving the main theorem.

Combining the upper and lower bounds for  $|\ell_{a,b}(c) \cap B|$  and the estimates for the number of  $c$  for which these intersections are non-empty yield that

$$r^2 \frac{\psi(h)}{h^{1-1/p}} = rh^{1/p} \cdot r \frac{\psi(h)}{h} \ll |\ell_{a,b} \cap B| \ll rh^{1/p} \cdot r \frac{\psi(h)}{h} = r^2 \frac{\psi(h)}{h^{1-1/p}}.$$

In other words, for any  $(a, b) \in \mathbb{N}^2$  and any open ball  $B \subset \Omega$  we have that

$$|\ell_{a,b} \cap B| \asymp |B| \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}}. \quad (14)$$

Thus, condition (10) holds for the sequence of sets  $\ell_{a,b} \cap B$ .

**2.4. Estimating the measure of  $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$ .** All that remains is to establish condition (11). Fix two distinct pairs of integers  $(a_1, b_1)$  and  $(a_2, b_2)$  in  $\mathcal{N}_{n,m}$  and for ease of notation set  $h_1 := h_{a_1,b_1}$  and  $h_2 := h_{a_2,b_2}$ . Recall that for  $(a, b) \in \mathcal{N}_{n,m}$  and  $c \in \mathbb{Z}_{\geq 0}$  our notation

$$R_{a,b}(c) := \{(x, y) \in [\epsilon, 1]^2 : a^n x + b^m y - c^p = 0\}.$$

A consequence of the assumption that our natural numbers  $a$  and  $b$  are coprime is that for any  $c_1$  and  $c_2$  the line segments  $R_{a_1,b_1}(c_1)$  and  $R_{a_2,b_2}(c_2)$  cannot be parallel. As we will see, this ensures that the intersection of the strips  $\ell_{a_1,b_1}$  and  $\ell_{a_2,b_2}$  is not too large. For the remainder of the section we denote by  $\alpha := \alpha(a_1, b_1, a_2, b_2)$  the strictly positive acute angle between the lines defining  $R_{a_1,b_1}(c_1)$  and  $R_{a_2,b_2}(c_2)$ ; that is, the angle between the vectors  $(a_1^n, b_1^m)$  and  $(a_2^n, b_2^m)$  in  $\mathbb{R}^2$ .

Given a ball  $B$  in  $[\epsilon, 1]^2$  we wish to deduce bounds on the size of the intersection  $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$ . To do this, we first fix an integer  $c_2$  and then estimate the size of each intersection  $\ell_{a_1,b_1} \cap \ell_{a_2,b_2}(c_2) \cap B$ . Firstly, observe that the set  $\ell_{a_2,b_2}(c_2) \cap B$  can be covered by a strip of length  $2r$  and of width  $\frac{2\psi(h)}{\sqrt{a_2^{2n} + b_2^{2m}}}$ . This strip is a section of the  $\frac{\psi(h)}{\sqrt{a_2^{2n} + b_2^{2m}}}$ -neighbourhood of the line  $a_2^n x + b_2^m y - c_2^p = 0$ . We now consider the size of the intersection of  $\ell_{a_1,b_1}$  with such a strip.

The set  $\ell_{a_1,b_1}$  consists of a collection of neighbourhoods of parallel lines of the form

$$a_1^n x + b_1^m y - c_1^p = 0, \quad c_1 \geq 0,$$

in  $[\epsilon, 1]^2$ ; that is, neighbourhoods of line segments  $R_{a_1,b_1}(c_1)$  for  $c_1 \geq 0$ . A simple geometric argument combined with the binomial theorem shows that the distance between any two such adjacent line segments, say  $R_{a_1,b_1}(c_1)$  and  $R_{a_1,b_1}(c_1 + 1)$ , is given by

$$\begin{aligned} \frac{\left(\frac{(c_1+1)^p - c_1^p}{b_1^m}\right) \left(\frac{(c_1+1)^p - c_1^p}{b_1^m}\right)}{\sqrt{\left(\frac{(c_1+1)^p - c_1^p}{b_1^m}\right)^2 + \left(\frac{(c_1+1)^p - c_1^p}{b_1^m}\right)^2}} &= \frac{((c_1+1)^p - c_1^p)((c_1+1)^p - c_1^p)}{\sqrt{a_1^{2n}((c_1+1)^p - c_1^p)^2 + b_1^{2m}((c_1+1)^p - c_1^p)^2}} \\ &\asymp \frac{(c_1+1)^p - c_1^p}{h_1} \asymp \frac{c_1^{p-1}}{h_1}. \end{aligned}$$

By the set of inequalities (12) we know that for the intersection  $\ell_{a_1,b_1} \cap B$  to be non-empty that  $c_1 \asymp h_1^{1/p}$ , and so this distance is  $\asymp h_1^{\frac{p-1}{p}-1} = h_1^{-1/p}$ . Therefore, if two adjacent line segments of the form  $R_{a_1,b_1}(c_1)$  intersect the line segment  $R_{a_2,b_2}(c_2)$  then the distance between the two intersection points is  $\asymp (h_1^{1/p} \sin \alpha)^{-1}$ . In turn, this implies that there are  $\ll rh_1^{1/p} \sin \alpha + 1$  non-empty intersections of this type for each fixed natural number  $c_2$ .

Furthermore, for each  $c_1$  the set  $\ell_{a_1, b_1}(c_1) \cap \ell_{a_2, b_2}(c_2)$  takes the form of a parallelepiped with volume  $\asymp \frac{\psi(h_1)\psi(h_2)}{h_1 h_2 \sin \alpha}$ . It follows that

$$|\ell_{a_1, b_1} \cap \ell_{a_2, b_2}(c_2) \cap B| \ll \frac{(r h_1^{1/p} \sin \alpha + 1) \psi(h_1) \psi(h_2)}{h_1 h_2 \sin \alpha}.$$

Finally, we recall from §2.3 that there are  $\asymp r h_2^{1/p}$  possible choices of  $c_2$  for which  $\ell_{a_2, b_2}(c_2) \cap B \neq \emptyset$ , and so

$$\begin{aligned} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| &\ll \frac{(r h_1^{1/p} \sin \alpha + 1) \psi(h_1) \psi(h_2) \cdot r h_2^{1/p}}{h_1 h_2 \sin \alpha} \\ &\asymp |B| \frac{\psi(h_1) \psi(h_2)}{h_1^{1-1/p} h_2^{1-1/p}} \cdot \left(1 + \frac{1}{r h_1^{1/p} \sin \alpha}\right). \end{aligned} \quad (15)$$

We now split our calculation into three exhaustive subcases depending on the size of the angle  $\alpha$ .

2.4.1. *Large angle.* First, we assume that  $\alpha$  is large enough to satisfy

$$\sin \alpha \geq \frac{1}{r h_1^{1/p}}. \quad (16)$$

It immediately follows from (14) and (15) that

$$|\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll |B| \frac{\psi(h_1) \psi(h_2)}{h_1^{1-1/p} h_2^{1-1/p}},$$

and so

$$\begin{aligned} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (16)} \\ h_1 \leq H, h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| &\ll |B| \sum_{\substack{(a_1, b_1) \in \mathcal{N}_{n, m} \\ h_1 \leq H}} \frac{\psi(h_1)}{h_1^{1-1/p}} \sum_{\substack{(a_2, b_2) \in \mathcal{N}_{n, m} \\ h_2 \leq H}} \frac{\psi(h_2)}{h_2^{1-1/p}} \\ &\asymp \frac{1}{|B|} \left( \sum_{(a, b) \in \mathcal{N}_{n, m}, h_{a, b} \leq H} |\ell_{a, b} \cap B| \right)^2. \end{aligned}$$

Thus, the set of pairs  $(a_1, b_1), (a_2, b_2) \in \mathcal{N}_{n, m}$  with property (16) satisfy condition (11).

2.4.2. *Medium angle.* We next consider the case when the angle between the line segments  $R_{a_1, b_1}(c_1)$  and  $R_{a_2, b_2}(c_2)$  is small, but not too small. In this case the intersection  $\ell_{a_1, b_1}(c_1) \cap \ell_{a_2, b_2}(c_2) \cap B$  may be quite large and contribute more than its fair share to the volume sum we are calculating. However, we show that the number of pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  satisfying both this property is sufficiently small. To be precise, we now assume that

$$\frac{1}{r h_1^{1/p}} \geq \sin \alpha \geq \frac{1}{r^{1+\frac{k}{M}} h_1^{\frac{k}{pN}} h_2^{\frac{1}{p}}}, \quad (17)$$

where, for ease of notation we have set  $N := \max(n, m)$ ,  $M := \min(n, m)$  and  $k = \gcd(n, m)$ . It immediately follows from equations (15) and (17) that

$$|\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll r \frac{\psi(h_1) \psi(h_2)}{h_1^{1-\frac{1}{p}} h_2 \sin \alpha} \leq r^{2+\frac{k}{M}} \frac{\psi(h_1) \psi(h_2)}{h_1^{1-\frac{1}{p}-\frac{k}{pN}} h_2^{1-\frac{1}{p}}}. \quad (18)$$

We now find an upper bound for the number of quadruples  $(a_1, b_1, a_2, b_2)$  satisfying (17). In what follows, we denote by  $\|M\|$  the absolute value of the determinant of a matrix  $M$ . To begin with, observe that the natural numbers  $a_1, b_1, a_2$  and  $b_2$  satisfy

$$a_1^n b_2^m - a_2^n b_1^m = \left( a_1^{\frac{n}{k}} b_2^{\frac{m}{k}} - a_2^{\frac{n}{k}} b_1^{\frac{m}{k}} \right) \sum_{t=1}^k a_1^{n(1-\frac{t}{k})} b_1^{\frac{m(t-1)}{k}} a_2^{\frac{n(t-1)}{k}} b_2^{m(1-\frac{t}{k})}.$$

For  $k = 1$  (and  $n = m = 1$ ) this statement is trivial and the sum on the right hand side is simply 1, so for now assume that  $k \geq 2$ . In view of the defining properties of the set  $\mathcal{N}_{n,m}$ , for  $i = 1, 2$  and for any  $t = 2, \dots, k$  we have

$$\left( \frac{1}{2} \right)^{1-1/k} \leq \left( \frac{1}{2} \right)^{(t-1)/k} \leq \frac{a_i^{n(t-1)/k}}{b_i^{m(t-1)/k}} \leq 2^{1-1/k} \leq 2^{1-1/k}$$

and for  $t = 1, \dots, k-1$  we have

$$\left( \frac{1}{2} \right)^{2-1/k} \leq \left( \frac{1}{2} \right)^{1+t/k} \leq \frac{a_i^{n(1-t/k)}}{b_i^{m(1-t/k)}} \leq 2^{1+t/k} \leq 2^{2-1/k}.$$

It follows that

$$\begin{aligned} \sum_{t=1}^k a_1^{n(1-t/k)} b_1^{m(t-1)/k} a_2^{n(t-1)/k} b_2^{m(1-t/k)} &\asymp \sum_{t=1}^k h_1^{(1-t/k)+(t-1)/k} h_2^{(1-t/k)+(t-1)/k} \\ &= \sum_{t=1}^k h_1^{1-1/k} h_2^{1-1/k} \asymp h_1^{1-1/k} h_2^{1-1/k}. \end{aligned}$$

Furthermore, since  $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$  it is certain that  $a_1^{\frac{n}{k}} b_2^{\frac{m}{k}} - a_2^{\frac{n}{k}} b_1^{\frac{m}{k}}$  is non-zero. As  $\alpha$  is precisely the positive (acute) angle between the two vectors  $(a_1^n, b_1^m)$  and  $(a_2^n, b_2^m)$ , the cross product formula readily implies that

$$\begin{aligned} 1 &\leq \left\| \begin{pmatrix} a_1^{n/k} & b_1^{m/k} \\ a_2^{n/k} & b_2^{m/k} \end{pmatrix} \right\| \asymp h_1^{1/k-1} h_2^{1/k-1} \left\| \begin{pmatrix} a_1^n & b_1^m \\ a_2^n & b_2^m \end{pmatrix} \right\| \\ &= h_1^{1/k-1} h_2^{1/k-1} \sin \alpha \sqrt{a_1^{2n} + b_1^{2m}} \sqrt{a_2^{2n} + b_2^{2m}} \\ &\asymp h_1^{1/k} h_2^{1/k} \sin \alpha. \end{aligned} \tag{19}$$

One can easily verify that this system of inequalities also holds in the case  $n = m = k = 1$ . In particular, for any  $k$ , assume that we have two pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  both in  $\mathcal{N}_{n,m}$  and both satisfying (17). It follows that  $1 \leq \left| a_1^{\frac{n}{k}} b_2^{\frac{m}{k}} - a_2^{\frac{n}{k}} b_1^{\frac{m}{k}} \right| \leq r^{-1} h_1^{\frac{1}{k}-\frac{1}{p}} h_2^{\frac{1}{k}}$ . Therefore,

$$\left| b_1^{\frac{m}{k}} - a_2^{-\frac{n}{k}} a_1^{\frac{n}{k}} b_2^{\frac{m}{k}} \right| \leq r^{-1} a_2^{-\frac{n}{k}} h_1^{\frac{1}{k}-\frac{1}{p}} h_2^{\frac{1}{k}} \ll r^{-1} h_1^{\frac{1}{k}-\frac{1}{p}}.$$

Now, assume that the pair  $(a_2, b_2)$ , and therefore  $h_2$ , is fixed. Then the above calculation implies that for each fixed  $a_1$  there are at most a constant times  $r^{-k/m} h_1^{(1-k/p)/m}$  possible choices for  $b_1$ . Similarly, if one were to fix  $b_1$  then we would have at most a constant times  $r^{-k/n} h_1^{(1-k/p)/n}$  ways of choosing  $a_1$ . Without loss of generality we will assume that these quantities are greater than 1 for the remainder of this subsection.

We now calculate the total volume that the intersections  $\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B$  contribute to our measure sum in the case that (17) holds. For conciseness of notation define the



function  $f : \mathbb{N}^2 \times \{n, m\} \rightarrow \mathbb{N}$  in the following way:

$$f(a, b, i) = \begin{cases} a^n, & i = n. \\ b^m, & i = m. \end{cases}$$

Without loss of generality we assume that  $h_2 \geq h_1$ . Following on from (18) we have that

$$\begin{aligned} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 \leq h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| &\ll r^{2 + \frac{k}{M}} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 \leq h_2 \leq H}} \frac{\psi(h_1)\psi(h_2)}{h_1^{1 - \frac{1}{p} - \frac{k}{pN}} h_2^{1 - \frac{1}{p}}} \\ &\leq r^{2 + \frac{k}{M}} (S_{n, n} + S_{n, m} + S_{m, n} + S_{m, m}), \end{aligned}$$

where

$$S_{i, j} := \sum_{g_2=1}^{\lfloor H^{1/j} \rfloor} \sum_{g_1=1}^{\lfloor g_2^{j/i} \rfloor} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = f(a_1, b_1, i) = g_1^i, \\ h_2 = f(a_2, b_2, j) = g_2^j}} \frac{\psi(h_1)\psi(h_2)}{h_1^{1 - \frac{1}{p} - \frac{k}{pN}} h_2^{1 - \frac{1}{p}}}.$$

The somewhat cumbersome collection of sum conditions in the final expression serve a very important purpose. We have split our volume calculation into four smaller sums  $S_{i, j}$  (for  $i, j \in \{n, m\}$ ). Each sum corresponds to the pairs of natural numbers  $(a_1, b_1)$  and  $(a_2, b_2)$  for which  $h_1$  takes the form of an  $i$ 'th power and  $h_2$  takes the form of a  $j$ 'th power. However, it is for example perfectly possible for the height  $h_1$  to take the form of an  $n$ -th power yet to be attained on the second component; i.e., we could have for some natural number  $b$  that  $a_1^{n/m} < b_1 = b^n$  and so  $h_1 = b_1^m = (b^n)^m = (b^m)^n$ . If  $(a_1, b_1)$  does take this form then we do not wish to count in the sums  $S_{n, n}$  or  $S_{n, m}$  the quadruple  $(a_1, b_1, a_2, b_2)$  as they will have already appeared in the sum  $S_{m, n}$  if  $h_2 = a_2^n$  or  $S_{m, m}$  if  $h_2 = b_2^m$ . The function  $f$  guarantees that the values taken by  $h_1$  and  $h_2$  genuinely appear in the first component if they appear as  $g_1^n$  or  $g_2^n$  and the second component if they appear as  $g_1^m$  or  $g_2^m$ . This painstaking stipulation will prove important in our calculation.

For the purpose of clarity we will exhibit the calculations relating to the two sums  $S_{n, n}$  and  $S_{m, n}$  separately; upper bounds for the remaining sums  $S_{m, m}$  and  $S_{n, m}$  follow analogously.

Firstly, the case  $i = j = n$  corresponds to those pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  for which  $h_1 = a_1^n$  and  $h_2 = a_2^n$ . We are fairly powerless but to use the crude upper bound of  $h_2^{1/m}$  to estimate the number of possible pairs  $(a_2, b_2)$  satisfying  $h_2 = a_2^n$  - it turns out that taking into account conditions (6) to reduce this trivial bound serves little purpose here. However, once  $a_1$  is also chosen we may use our estimate for the number of ways of choosing the integer  $b_1$  so that (17) holds once the other three natural numbers  $a_1, a_2, b_2$  are already prescribed. For fixed natural numbers  $g_1$  and  $g_2$  with  $g_1 \leq g_2$  we have

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = a_1^n = g_1^n, \\ h_2 = a_2^n = g_2^n}} 1 \leq g_2^{\frac{n}{m}} \cdot r^{-\frac{k}{m}} g_1^{\frac{n}{m}(1 - \frac{k}{p})}.$$

Therefore,

$$\begin{aligned}
\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = a_1^n = g_1^n, \\ h_2 = a_2^n = g_2^n}} \frac{\psi(h_1)\psi(h_2)}{h_1^{1-\frac{1}{p}-\frac{k}{pN}} h_2^{1-\frac{1}{p}}} &= \frac{\psi(g_1^n)\psi(g_2^n)}{g_1^{n(1-\frac{1}{p}-\frac{k}{pN})} g_2^{n(1-\frac{1}{p})}} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = a_1^n = g_1^n, \\ h_2 = a_2^n = g_2^n}} 1 \\
&\leq r^{-\frac{k}{m}} \frac{\psi(g_1^n)\psi(g_2^n)}{g_1^{n(1-\frac{1}{p}-\frac{1}{m}-\frac{k}{pN}+\frac{k}{pm})} g_2^{n(1-\frac{1}{p}-\frac{1}{m})}} \\
&\leq r^{-\frac{k}{M}} \frac{\psi(g_1^n)}{g_1^{n(1-\frac{1}{p}-\frac{1}{m})}} \frac{\psi(g_2^n)}{g_2^{n(1-\frac{1}{p}-\frac{1}{m})}},
\end{aligned}$$

and so  $S_{n, n}$  is bounded above by

$$r^{-\frac{k}{M}} \sum_{g_2=1}^{\lfloor H^{1/n} \rfloor} \sum_{g_1=1}^{g_2} \frac{\psi(g_1^n)}{g_1^{n(1-\frac{1}{p}-\frac{1}{m})}} \frac{\psi(g_2^n)}{g_2^{n(1-\frac{1}{p}-\frac{1}{m})}} \leq r^{-\frac{k}{M}} \left( \sum_{a=1}^{\lfloor H^{1/n} \rfloor} \frac{\psi(a^n)}{a^{n(1-\frac{1}{p}-\frac{1}{m})}} \right)^2.$$

Next, in the case  $i = m, j = n$  the sum  $S_{m, n}$  corresponds to those pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  for which  $h_1 = b_1^m$  and  $h_2 = a_2^n$ . Here we use our estimate for the number of ways of choosing the integer  $a_1$  so that (17) is satisfied once the other three natural numbers  $b_1, a_2, b_2$  are already chosen. For any given natural numbers  $g_1$  and  $g_2$  with  $g_1 \leq g_2$  we have

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = b_1^m = g_1^m, \\ h_2 = a_2^n = g_2^n}} 1 \leq g_2^{\frac{n}{m}} \cdot r^{-\frac{k}{n}} g_1^{\frac{m}{n}(1-\frac{k}{p})}.$$

As before it follows that

$$\begin{aligned}
\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = b_1^m = g_1^m, \\ h_2 = a_2^n = g_2^n}} \frac{\psi(h_1)\psi(h_2)}{h_1^{1-\frac{1}{p}-\frac{k}{pN}} h_2^{1-\frac{1}{p}}} &= \frac{\psi(g_1^m)\psi(g_2^n)}{g_1^{m(1-\frac{1}{p}-\frac{k}{pN})} g_2^{n(1-\frac{1}{p})}} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n, m} \\ \text{satisfying (17)} \\ h_1 = b_1^m = g_1^m, \\ h_2 = a_2^n = g_2^n}} 1 \\
&\leq r^{-\frac{k}{n}} \frac{\psi(g_1^m)\psi(g_2^n)}{g_1^{m(1-\frac{1}{p}-\frac{1}{n}-\frac{k}{pN}+\frac{k}{pn})} g_2^{n(1-\frac{1}{p}-\frac{1}{m})}} \\
&\leq r^{-\frac{k}{M}} \frac{\psi(g_1^m)}{g_1^{m(1-\frac{1}{p}-\frac{1}{n})}} \frac{\psi(g_2^n)}{g_2^{n(1-\frac{1}{p}-\frac{1}{m})}},
\end{aligned}$$

and so

$$S_{m, n} \leq r^{-\frac{k}{M}} \left( \sum_{b=1}^{\lfloor H^{1/m} \rfloor} \frac{\psi(b^m)}{b^{m(1-\frac{1}{p}-\frac{1}{n})}} \right) \left( \sum_{a=1}^{\lfloor H^{1/n} \rfloor} \frac{\psi(a^n)}{a^{n(1-\frac{1}{p}-\frac{1}{m})}} \right).$$

One can carry out analogous calculations corresponding to the remaining two cases, leading to the estimates

$$S_{n, m} \leq r^{-\frac{k}{M}} \left( \sum_{a=1}^{\lfloor H^{1/n} \rfloor} \frac{\psi(a^n)}{a^{n(1-\frac{1}{p}-\frac{1}{m})}} \right) \left( \sum_{b=1}^{\lfloor H^{1/m} \rfloor} \frac{\psi(b^m)}{b^{m(1-\frac{1}{p}-\frac{1}{n})}} \right)$$

and

$$S_{m,m} \leq r^{-\frac{k}{M}} \left( \sum_{b=1}^{\lfloor H^{1/m} \rfloor} \frac{\psi(b^m)}{b^{m(1-\frac{1}{p}-\frac{1}{n})}} \right)^2.$$

Thus, the sum of sums  $S_{n,n} + S_{n,m} + S_{m,n} + S_{m,m}$  is bounded above by

$$r^{-\frac{k}{M}} \left( \sum_{a=1}^{\lfloor H^{1/n} \rfloor} \frac{\psi(a^n)}{a^{n(1-\frac{1}{p}-\frac{1}{m})}} + \sum_{b=1}^{\lfloor H^{1/m} \rfloor} \frac{\psi(b^m)}{b^{m(1-\frac{1}{p}-\frac{1}{n})}} \right)^2 \asymp r^{-\frac{k}{M}} \left( \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b} \leq H}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-\frac{1}{p}}} \right)^2.$$

It follows from §2.2.2 that this quantity is in turn bounded above by a constant times

$$\frac{1}{r^{4+k/M}} \left( \sum_{\substack{(a,b) \in \mathcal{N}_{n,m} \\ h_{a,b} \leq H}} |B| \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} \right)^2 \asymp \frac{1}{r^{4+k/M}} \left( \sum_{\substack{(a,b) \in \mathcal{N}_{n,m} \\ h_{a,b} \leq H}} |\ell_{a,b} \cap B| \right)^2,$$

and so the total contribution to our volume sum made by pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  for which (17) holds must satisfy

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n,m} \\ \text{satisfying (17)} \\ h_1 \leq h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll \frac{1}{|B|} \left( \sum_{\substack{(a,b) \in \mathcal{N}_{n,m} \\ h_{a,b} \leq H}} |\ell_{a,b} \cap B| \right)^2.$$

Condition (11) therefore holds in the medium angle case.

**2.4.3. Small angle.** Finally, we consider the scenario in which the angle  $\alpha$  between the vectors  $(a_1^n, b_1^m)$  and  $(a_2^n, b_2^m)$  is very small indeed; that is, we assume that

$$\sin \alpha < r^{-(1+\frac{k}{M})} h_1^{-\frac{k}{pN}} h_2^{-\frac{1}{p}}. \quad (20)$$

Whilst the intersection inside the ball  $B$  of the corresponding strips  $\ell_{a_1, b_1}$  and  $\ell_{a_2, b_2}$  may now be very large due to the strips being close to parallel, we show that since  $(a, b)$  are chosen only from  $\mathcal{N}_{n,m}$  then there are very few pairs indeed satisfying this condition. In fact, we show there are so few that the total volume contributed by intersections of this type is finite.

Recall that inequality (19) tells us that any two distinct pairs of natural numbers  $(a_1, b_1)$  and  $(a_2, b_2)$  in  $\mathcal{N}_{n,m}$  must also satisfy  $\sin \alpha \geq h_1^{-1/k} h_2^{-1/k}$ . It follows from (15) that

$$|\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll r \frac{\psi(h_1) \psi(h_2)}{h_1^{1-1/p} h_2 \sin \alpha} \leq r \frac{\psi(h_1) \psi(h_2)}{h_1^{1-1/p-1/k} h_2^{1-1/k}}. \quad (21)$$

To calculate the total volume contributed by intersections of strips  $\ell_{a_1, b_1}$  and  $\ell_{a_2, b_2}$  satisfying (20) we again specialise by splitting our total volume sum into four smaller sums depending on whether  $h_1$  and  $h_2$  take  $n$ -th powers or  $m$ -th powers. Unlike in the medium angle case we will not need to assume that the heights  $h_1$  and  $h_2$  are attained in the ‘correct component’ and so for the sake of simplicity will allow a certain amount of overlap in these sums. Once again, we assume without loss of generality that  $h_2 \geq h_1$  throughout.

It is readily verified from (21) that

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n,m} \\ \text{satisfying (20)} \\ h_1 \leq h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| \ll r (T_{n,n} + T_{n,m} + T_{m,n} + T_{m,m}), \quad (22)$$

where

$$T_{i,j} := \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{i}{t_1} t_2 \rfloor} \sum_{\substack{2^{t_1} < g_1 \leq 2^{t_1+1} \\ 2^{t_2} < g_2 \leq 2^{t_2+1}}} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n,m} \\ \text{satisfying (20)} \\ h_1 = g_1^i, h_2 = g_2^j}} \frac{\psi(h_1)\psi(h_2)}{h_1^{1-\frac{1}{p}-\frac{1}{k}} h_2^{1-\frac{1}{k}}}.$$

We will find an upper bound for all four sums  $T_{i,j}$  (for  $i, j \in \{n, m\}$ ) simultaneously, beginning with the observation that each

$$T_{i,j} \ll \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{i}{t_1} t_2 \rfloor} \frac{\psi(2^{it_1})\psi(2^{jt_2})}{2^{it_1(1-\frac{1}{p}-\frac{1}{k})} 2^{jt_2(1-\frac{1}{k})}} \sum_{\substack{2^{t_1} < g_1 \leq 2^{t_1+1} \\ 2^{t_2} < g_2 \leq 2^{t_2+1}}} \sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathbb{N}^2 \\ \text{satisfying (20)} \\ h_1 = g_1^i, h_2 = g_2^j}} 1.$$

To deduce an estimate for the final two sums in the above expression we will utilise the following. By combining inequality (19) with (20) we deduce that

$$1 \leq \left\| \begin{pmatrix} a_1^{\frac{n}{k}} & b_1^{\frac{m}{k}} \\ a_2^{\frac{n}{k}} & b_2^{\frac{m}{k}} \end{pmatrix} \right\| \leq h_1^{\frac{1}{k}} h_2^{\frac{1}{k}} \sin \alpha < \frac{h_1^{\frac{1}{k}-\frac{k}{pN}} h_2^{\frac{1}{k}-\frac{1}{p}}}{r^{1+\frac{k}{M}}}. \quad (23)$$

For each of our fixed indices  $t_1, t_2$  we wish to count the number of quadruples  $(a_1, b_1, a_2, b_2)$  satisfying (6), (20) and

$$h_1 = g_1^i, \quad h_2 = g_2^j, \quad 2^{t_1} < g_1 \leq 2^{t_1+1}, \quad 2^{t_2} < g_2 \leq 2^{t_2+1}. \quad (24)$$

We begin with the observation that there are at most  $2^{jt_2/n}$  possible choices for  $a_2$  satisfying (24) and  $2^{it_1/m}$  ways of choosing  $b_1$ . Given fixed  $a_2, b_1$  then (23) implies that there are  $\ll r^{1+\frac{k}{M}} h_1^{\frac{1}{k}-\frac{k}{pN}} h_2^{\frac{1}{k}-\frac{1}{p}}$  possible values for the integer  $l = a_1^{\frac{n}{k}} b_2^{\frac{m}{k}}$ . Denoting by  $d$  the divisor function, for each such natural number  $l$  there are at most  $d(l)$  possible values for the pair  $a_1^{n/k}$  and  $b_2^{m/k}$ , and so at most  $d(l)$  possible values for the pair  $a_1$  and  $b_2$ . Once either  $a_1$  and  $b_2$  has been chosen. It is well known (see for example [17, Thm 7.2]) that for any  $\delta > 0$  there is a constant  $c_\delta$  such that  $d(t) \leq c_\delta t^\delta$  for all  $t \geq 1$  and so the number of ways of choosing  $l$  is  $\ll l^\delta \ll 2^{\frac{i}{k}t_1} 2^{\frac{j}{k}t_2}$ . We will fix a suitable value for  $\delta$  later in the calculation, and for now note that for any fixed  $\delta$  the number of possible quadruples satisfying  $(a_1, b_1, a_2, b_2)$  satisfying (6), (20) and (24) is

$$\ll 2^{\frac{i}{m}t_1} 2^{\frac{j}{n}t_2} \frac{\left(2^{it_1(\frac{1}{k}-\frac{k}{pN})} 2^{jt_2(\frac{1}{k}-\frac{1}{p})}\right)}{r^{1+\frac{k}{M}}} \left(2^{\frac{i}{k}t_1} 2^{\frac{j}{k}t_2}\right)^\delta = \frac{2^{it_1(\frac{1}{m}+\frac{1}{k}-\frac{k}{pN}+\frac{\delta}{k})} 2^{jt_2(\frac{1}{n}+\frac{1}{k}-\frac{1}{p}+\frac{\delta}{k})}}{r^{1+\frac{k}{M}}}.$$

An upper bound for each  $T_{i,j}$  is therefore given by

$$\begin{aligned}
T_{i,j} &\ll \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{j}{i} t_2 \rfloor} \frac{\psi(2^{it_1})\psi(2^{jt_2})}{2^{it_1(1-\frac{1}{p}-\frac{1}{k})} 2^{jt_2(1-\frac{1}{k})}} \cdot \frac{2^{it_1(\frac{1}{m}+\frac{1}{k}-\frac{k}{pN}+\frac{\delta}{k})} 2^{jt_2(\frac{1}{n}+\frac{1}{k}-\frac{1}{p}+\frac{\delta}{k})}}{r^{1+\frac{k}{M}}} \\
&= \frac{1}{r^{1+k/M}} \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{j}{i} t_2 \rfloor} \psi(2^{it_1})\psi(2^{jt_2}) 2^{it_1(\frac{\delta+2}{k}+\frac{1}{m}+\frac{1}{p}-\frac{k}{pN}-1)} 2^{jt_2(\frac{\delta+2}{k}+\frac{1}{n}-\frac{1}{p}-1)} \\
&\leq \frac{1}{r^{1+k/M}} \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{j}{i} t_2 \rfloor} \psi(2^{it_1})\psi(2^{jt_2}) 2^{it_1(\frac{\delta+2}{k}+\frac{1}{m}-\frac{k}{2pN}-1)} 2^{jt_2(\frac{\delta+2}{k}+\frac{1}{n}-\frac{k}{2pN}-1)}.
\end{aligned}$$

The final inequality follows by the fact that  $it_1 \leq jt_2$  and  $k \leq N$ , so

$$2^{it_1(\frac{1}{p}-\frac{k}{2pN})} \leq 2^{jt_2(\frac{1}{p}-\frac{k}{2pN})}.$$

Now, since we are assuming  $\psi$  is monotonic and the sum

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ h_{a,b} \leq H}} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} \asymp \sum_{a \leq H^{1/n}} \frac{\psi(a^n)}{a^{n(1-1/p-1/m)}} + \sum_{b \leq H^{1/m}} \frac{\psi(b^m)}{b^{m(1-1/p-1/n)}} \quad (25)$$

diverges as  $H \rightarrow \infty$ , we may without loss of generality assume that  $\psi(q) \leq q^{-(\frac{1}{p}+\frac{1}{N})}$  for all sufficiently large natural numbers  $q$ . If this were not true for our function  $\psi$  then we could simply set  $\psi_0(q) := \min(\psi(q), q^{-(1/p+1/N)})$ . For, it is clear that the sum on the left hand side of (25) still diverges upon replacing  $\psi$  with  $\psi_0$  and that we have  $V_{n,m}^p(\psi_0) \subseteq V_{n,m}^p(\psi)$ . It follows that in order to prove Theorem 1 it would suffice to verify that  $|V_{n,m}^p(\psi_0)| = 1$ . With this in mind, we deduce that for  $H$  large enough

$$\begin{aligned}
T_{i,j} &\ll \frac{1}{r^{1+k/M}} \sum_{t_2=1}^{\lfloor \log(H^{1/j}) \rfloor} \sum_{t_1=1}^{\lfloor \frac{j}{i} t_2 \rfloor} 2^{it_1(\frac{\delta+2}{k}+\frac{1}{m}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-1)} 2^{jt_2(\frac{\delta+2}{k}+\frac{1}{n}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-1)} \\
&\leq \frac{1}{r^{1+k/M}} \sum_{t_2=1}^{\infty} 2^{jt_2(\frac{\delta+2}{k}+\frac{1}{n}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-1)} \sum_{t_1=1}^{\infty} 2^{it_1(\frac{\delta+2}{k}+\frac{1}{m}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-1)} \\
&\asymp \frac{1}{r^{1+k/M}} \sum_{a=1}^{\infty} a^{j(\frac{\delta+2}{k}+\frac{1}{n}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-\frac{1}{j}-1)} \sum_{b=1}^{\infty} b^{i(\frac{\delta+2}{k}+\frac{1}{m}-\frac{k}{2pN}-\frac{1}{p}-\frac{1}{N}-\frac{1}{i}-1)},
\end{aligned}$$

by Cauchy condensation. The final two sums both converge for every choice of  $i$  and  $j$  so long as we can choose  $\delta$  satisfying

$$\begin{aligned}
0 < \delta &< k \left( 1 + \frac{1}{p} + \frac{1}{N} + \frac{k}{2pN} - \max \left( \frac{1}{m}, \frac{1}{n} \right) \right) - 2 \\
&= k \left( 1 + \frac{1}{p} + \frac{1}{N} + \frac{k}{2pN} - \frac{1}{M} \right) - 2.
\end{aligned} \quad (26)$$

Assuming (26) holds, when combined with (21) the above calculation yields that

$$\sum_{\substack{(a_1, b_1) \neq (a_2, b_2) \in \mathcal{N}_{n,m} \\ \text{satisfying (20)} \\ h_1 \leq h_2 \leq H}} |\ell_{a_1, b_1} \cap \ell_{a_2, b_2} \cap B| < \infty,$$

from which it follows that the quantity

$$\limsup_{H \rightarrow \infty} \left( \sum_{\substack{(a,b) \in \mathcal{N}_{n,m}, \ h_{a,b} \leq H \\ \text{satisfying (20)}}} |\ell_{a,b} \cap B| \right)^2 \left( \sum_{\substack{(a_1,b_1) \neq (a_2,b_2) \in \mathcal{N}_{n,m} \\ \text{satisfying (20), } h_1 \leq h_2 \leq H}} |\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B| \right)^{-1}$$

is zero and does not contribute to the measure of the set  $V_{n,m}^p(\psi) \cap B$ .

All that remains is to check that we may choose  $\delta$  satisfying condition (26). When  $n = m = k = p = 1$  this is trivially true. Otherwise, when  $k \geq 2$  notice that

$$k \left( 1 + \frac{1}{p} + \frac{1}{N} + \frac{k}{2pN} - \frac{1}{M} \right) - 2 \geq 2 \left( \frac{1}{p} + \frac{1}{N} + \frac{1}{pN} - \frac{1}{M} \right).$$

In particular, if  $n = m$  (and so  $N = M$ ) then the latter quantity is strictly positive for any choice of  $p$  and we are done. Otherwise, recall that according to observation (5) we may assume that

$$p \leq \frac{nm}{nm - n - m} = \frac{NM}{NM - N - M}.$$

Since  $n \neq m$  and  $\gcd(n, m) \geq 2$  we must have  $N > 2$  and  $M \geq 2$ , from which it follows that

$$\begin{aligned} 2 \left( \frac{1}{p} + \frac{1}{N} + \frac{1}{pN} - \frac{1}{M} \right) &\geq 2 \left( \frac{NM - N - M}{NM} + \frac{1}{N} + \frac{NM - N - M}{N^2M} - \frac{1}{M} \right) \\ &= 2 \left( 1 + \frac{1}{N} - \frac{2}{M} - \frac{1}{NM} - \frac{1}{N^2} \right) \\ &\geq \frac{2}{N} \left( 1 - \frac{1}{2} - \frac{1}{N} \right) > 0, \end{aligned}$$

as required.

### 3. PROOF OF THEOREM 2

For completeness we give a very brief introduction to Hausdorff measures and dimension. For further details see [8]. Let  $F \subset \mathbb{R}^n$ . Then, for any  $\rho > 0$  a countable collection  $\{B_i\}$  of balls in  $\mathbb{R}^n$  with diameters  $\text{diam}(B_i) \leq \rho$  such that  $F \subset \bigcup_i B_i$  is called a  $\rho$ -cover of  $F$ . For a dimension function  $f$  define

$$\mathcal{H}_\rho^f(F) = \inf \sum_i f(\text{diam} B_i),$$

where the infimum is taken over all possible  $\rho$ -covers  $\{B_i\}$  of  $F$ . It is easy to see that  $\mathcal{H}_\rho^f(F)$  increases as  $\rho$  decreases and so approaches a limit as  $\rho \rightarrow 0$ . This limit could be zero or infinity, or take a finite positive value. Accordingly, the *Hausdorff  $f$ -measure*  $\mathcal{H}^f$  of  $F$  is defined to be

$$\mathcal{H}^f(F) = \lim_{\rho \rightarrow 0} \mathcal{H}_\rho^f(F).$$

It is easily verified that Hausdorff measure is monotonic and countably sub-additive, and that  $\mathcal{H}^s(\emptyset) = 0$ . Thus it is an outer measure on  $\mathbb{R}^n$ . In the case that  $f(r) = r^s$  ( $s > 0$ ), the measure  $\mathcal{H}^f$  is the usual  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s$ . For any subset  $F$  one can easily verify that there exists a unique critical value of  $s$  at which  $\mathcal{H}^s(F)$  ‘jumps’ from

infinity to zero. The value taken by  $s$  at this discontinuity is referred to as the *Hausdorff dimension of  $F$*  and is denoted by  $\dim F$ ; i.e.,

$$\dim F := \inf\{s \in \mathbb{R}^+ : \mathcal{H}^s(F) = 0\}.$$

When  $s$  is an integer,  $n$  say, then  $\mathcal{H}^n$  coincides up to universal constants with standard  $n$ -dimensional Lebesgue measure. In particular, a set  $E \subseteq \mathbb{R}^n$  is null/full with respect to  $\mathcal{H}^n$  if and only if it is null/full with respect to usual Lebesgue measure. Hausdorff  $s$ -measure, like Lebesgue measure, is preserved (up to a constant) by certain well behaved maps. In particular, if  $g : E \rightarrow F$  is a bi-Lipschitz bijection between two sets in Euclidean space then  $\mathcal{H}^s(E) \asymp \mathcal{H}^s(F)$ .

The proof of Theorem 2 is analogous to the proof of Theorem 3.2 in [3] and we will not give it in full detail. However, some subtleties occur, and we give a short outline of the proof with details where these are non-trivial.

For the convergence case the assumption that  $\psi$  is monotonic becomes irrelevant. The proof follows by simply using a standard covering argument, which depends on modifying the argument in §2.1 appropriately for Hausdorff measure.

In view of the divergence part of Theorem 1, proving the divergence part of Theorem 2 is now surprisingly easy due to the remarkable mass transference principle for linear forms developed by Beresnevich & Velani [4, 5]. This principle relies upon a ‘slicing’ technique and allows us to transfer statements about the Lebesgue measure of general limsup sets to statements concerning Hausdorff measures. We outline a specialised setup here, tailored to our needs. For consistency, we appeal to the following original notation.

Assume we are given a family  $\mathcal{R} = (R_\alpha)_{\alpha \in J}$  of lines  $R_\alpha \subset \mathbb{R}^2$  indexed by an infinite countable set  $J$ . For every  $\alpha \in J$  and real  $\delta \geq 0$  define the  $\delta$ -neighbourhood of  $R_\alpha$  by

$$\Delta(R_\alpha, \delta) := \left\{ \mathbf{x} \in \mathbb{R}^2 : \inf_{\mathbf{y} \in R_\alpha} |\mathbf{x} - \mathbf{y}| < \delta \right\}.$$

Next, assume we are given a non-negative, real-valued function  $\Upsilon$  on  $J$ :

$$\Upsilon : J \rightarrow \mathbb{R}^+ : \alpha \mapsto \Upsilon(\alpha) := \Upsilon_\alpha.$$

Furthermore, to ensure that  $\Upsilon_\alpha \rightarrow 0$  as  $\alpha$  runs through  $J$  it is assumed that for every  $\epsilon > 0$  the set  $\{\alpha \in J : \Upsilon_\alpha > \epsilon\}$  is finite. Finally, denote by

$$\Lambda(\Upsilon) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \in \Delta(R_\alpha, \Upsilon_\alpha) \text{ for infinitely many } \alpha \in J \right\}$$

the set of points lying in the respective  $\Upsilon_\alpha$ -neighbourhood of infinitely many of the lines  $R_\alpha$ .

**Theorem 3** (Beresnevich & Velani [5]). *Let  $\mathcal{R}$  and  $\Upsilon$  be given as above. Let  $V$  be a line in  $\mathbb{R}^2$  such that*

- (1)  $V \cap R_\alpha \neq \emptyset$  for all  $\alpha \in J$ , and
- (2)  $\sup_{\alpha \in J} \text{diam}(V \cap \Delta(R_\alpha, 1)) < \infty$ .

*Let  $f$  and  $g : r \mapsto g(r) := r^{-1}f(r)$  be dimension functions such that  $r^{-2}f(r)$  is monotonic and let  $\Omega$  be a ball in  $\mathbb{R}^2$ . Suppose for any ball  $B$  in  $\Omega$  we have*

$$\mathcal{H}^2(B \cap \Lambda(g(\Upsilon))) = \mathcal{H}^2(B).$$

*Then*

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B).$$

We will apply Theorem 1 to ensure that Theorem 3 may be applied. We consider the set  $\Lambda(g(\Upsilon))$ , where  $\Upsilon = \psi(h)/h$  for  $R_\alpha = R_{a,b}(c)$  and  $h = h_{a,b}$ . Up to a universal constant  $c > 0$  the set  $\Delta(R_\alpha, \Upsilon_\alpha)$  contains one of the strips

$$\ell_{a,b}(c) := \{(x, y) \in [0, 1]^2 : |a^n x + b^m y - c^p| < c\Upsilon_\alpha h_{a,b}\}.$$

Hence, to prove that the set  $\Lambda(g(\Upsilon))$  is full with respect to  $\mathcal{H}^2$ , we need only ensure that the series

$$\sum_{(a,b) \in \mathbb{N}^2} \frac{c\Upsilon_\alpha h_{a,b}}{h_{a,b}^{1-1/p}} = c \sum_{(a,b) \in \mathbb{N}^2} g\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right) h_{a,b}^{1/p}$$

diverges, which is exactly our assumption.

It remains for us to find a line  $V$  together with a restricted set of pairs  $(a, b)$  satisfying the assumptions of Theorem 3. However, in our proof of Theorem 1, we initially introduced the restriction (6), which implies that all lines  $R_\alpha$  considered have slope in the interval from  $-2$  to  $-1/2$ . Taking  $J = \{(a, b) \in \mathbb{N}^2 : \gcd(a, b) = 1, \frac{1}{2} \leq \frac{a^n}{b^m} \leq 2\}$  and  $V$  to be the line given by the equation  $y = x$  clearly gives us properties (1) and (2) from Theorem 3.

We now have all assumptions satisfied and may conclude that for any ball  $B$ ,

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B),$$

so that in particular

$$\mathcal{H}^f(\Lambda(\Upsilon) \cap [0, 1]^2) = \mathcal{H}^f([0, 1]^2).$$

It is now a simple matter to verify that  $\Lambda(\Upsilon) \cap [0, 1]^2 \subseteq W_{n,m}^p(\psi)$ , and Theorem 2 follows.

Finally, to deduce Corollaries 1 & 2 from Theorem 2, we use standard arguments such as those exhibited in the proofs of Corollaries 3.3 & 3.4 in [3].

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